Bifurcation and Stability Analysis for Environmental Pollutants through Three-layered Population and Rain

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Abstract

Globe is suffering from the environmental pollutants due to plant, animal and human. The environmental pollutants are increased in last few years due to different activities done by nature as well as humans. In this paper, we have taken three-layered population viz. plant population, animal population and human population affecting environment. A significant growth in environmental pollutants has decreased the intensity of rain throughout the world. The system of non-linear ordinary differential equations is formulated to analyse the proposed problem. Bifurcation and stability are discussed for environmental pollutants. Numerical simulation is carried out to study the effect of rain intensity due to three-layered population using data.

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1. Introduction

Rain is the set of water droplets which contain some amount of atmospheric water vapor and falls when it becomes heavy under gravitation force of Earth. Rain is the only natural source of water cycle and freshwater for living species. Earth's surface is dense with water and most of it is salty sea water which we can not drink and is useless to the most of organism. This is the reason why rain is crucial to life. All plants require at least some amount of water to survive; animal and human cells are made up from 90 percent of water. Therefore, it can be said that life could not be possible without rain water. But living things itself affect the environment either directly or indirectly. They create environmental pollution which is slowly diminishing rain. The same effect is observed with high impact in highly polluted areas in terms of rain intensity. Rain intensity is defined as the ratio of the total amount of rain which falls over a constant time period. While studies can not deliver suitable proof, strong intuition is that environmental pollutants reduce rain intensity said by Ulrike Lohmann. This motivates scientists to collect data to measure the effect between environmental pollutants and rain intensity.

Khemani and Murty surveyed rainfall variation in an urban industrial region in 1973. Shukla *et al.* observed effect of rain on removal of a gaseous pollutant and two different particulate matters from the atmosphere of a city in 2008. Shukla *et al.* also extended their model with the effect of cloud density after that in 2008. In the year 2017, Sharma and Kumari prepared modeling the impact of rain on population exposed to air pollution. Some researchers worked on environmental pollution. In 1992, Pandey et al. studied air pollutant concentrations in Varanasi, India. Dubey and Hussain in 2000 expressed modelling the interaction of two biological species in a polluted environment. Qualitative analysis of a nonlinear model for removal of air pollutants was analysed in 2003 by Naresh. Shukla et al. shown modeling effects of primary and secondary toxicants on renewable resources in 2003. A model for the effect of pollutant on human population dependent on a resource with environmental and health policy was developed by Dubey in 2010. Shah et al. in 2017 proposed optimum control for spread of pollutants through forest resources. Also, they formulated mathematical approach on household waste causing environmental pollutants due to landfill and treatments in 2018. Some models based on environment are developed through mathematics by some researchers. Optimal control on depletion of green belt due to industries was studied by Shah et al. in 2017. Kademi et al. formulated modelling the dynamics of toxicity associated with aflatoxins in foods and feeds in 2017. In 2018, Shah et al. computed stability of 'GO-CLEAN' model through graphs. In the proposed paper, a mathematical model is developed using some assumption with existence of equilibrium points and threshold quantity in section 2. In section 3, local and global stability analysis is discussed. The bifurcation analysis is computed in section 4. The model is validated in section 5 through numerical simulation.

2. Formulation of the model

In our model, we take five different compartments namely the animal density (A), the plant density (P), the human density (H), the environmental pollutants (E_P) and the rain intensity (R) to observe the effect of environmental pollutants – emitted through three population: Animal, Plant and Human – on rain. Figure 1 shows the transmission diagram of the environmental pollutants and table 1 gives information about the description and value of the parameters used in the model.

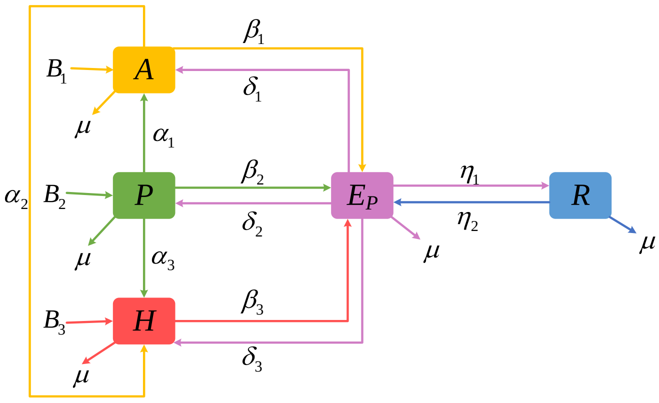

Figure 1: Transmission diagram of the rain model

Table 1: Notation, description and parametric values

Notation	Description	Parametric value
B_1	The growth rate of animal population	0.1
B_2	The growth rate of plant population	0.3
B_3	The growth rate of human population	0.5
β_1	The rate of environmental pollutants caused by dead animal	0.25
β_2	The rate of environmental pollutants caused by dead plant	0.1
β_3	The rate of environmental pollutants caused by human activities	0.4
α_1	The rate at which animal intakes plant as a food	0.5
α_2	The rate at which animal is taken as a food by human	0.01
α_3	The rate at which human intakes plant produce	0.7
δ_1	The rate of environmental pollutants absorbed by animal	0.15
δ_2	The rate of environmental pollutants absorbed by plant	0.3
δ_3	The rate of environmental pollutants absorbed by human	0.1
η_1	The rate at which environmental pollutants decrease rain intensity	0.05
η_2	The rate at which rain dissolves environmental pollutants	0.3
μ	The rate of unobserved activity through respective compartment	0.4

The following system of ordinary differential equations represents the model:

$$\frac{dA}{dt} = B_1 - \beta_1 A E_P + \delta_1 E_P + \alpha_1 P - \alpha_2 A - \mu A$$

$$\frac{dP}{dt} = B_2 - \beta_2 P E_P + \delta_2 E_P - \alpha_1 P - \alpha_3 P - \mu P$$

$$\frac{dH}{dt} = B_3 - \beta_3 H E_P + \delta_3 E_P + \alpha_2 A + \alpha_3 P - \mu H$$

$$\frac{dE}{dt} = \beta_1 A E_P + \beta_2 P E_P + \beta_3 H E_P - \delta_1 E_P - \delta_2 E_P - \delta_3 E_P - \eta_1 E_P + \eta_2 R - \mu E_P$$

$$\frac{dR}{dt} = \eta_1 E_P - \eta_2 R - \mu R$$

$$A + P + H + E_P + R \le N$$ and

$$A, P, H > 0; E_P, R \ge 0$$

(1)

2.1. Assumptions

1) Animal, plant and human population results environmental pollutants due to degradation, death at a constant rate.
2) The growth rate of human population is the highest.
3) Some of the environmental pollutants are soluble.
4) Environmental pollutants free equilibrium point of the proposed system will exist because of rain.
5) For simplicity, rate of the unobserved activity through respective compartment (μ) has been taken as constant.

2.2. Existence of equilibrium points

Solution of system of equation (1) gives equilibrium point when equations are set to zero. On solving system (1), we get two equilibrium points;

i. *Environmental pollutants free equilibrium point*

$E_0(A, P, H, 0, 0)$ where

$$A = \frac{B_1(\alpha_1 + \alpha_3 + \mu) + B_2\alpha_1}{(\alpha_1 + \alpha_3 + \mu)(\alpha_2 + \mu)}, \quad P = \frac{B_2}{\alpha_1 + \alpha_3 + \mu},$$

$$H = \frac{B_1\alpha_2(\alpha_1 + \alpha_3 + \mu) + B_2(\alpha_2(\alpha_1 + \alpha_3) + \alpha_3\mu) + B_3(\alpha_1 + \alpha_3 + \mu)(\alpha_2 + \mu)}{\mu(\alpha_1 + \alpha_3 + \mu)(\alpha_2 + \mu)}$$

This equilibrium point always exists without any condition.

ii. *Interior equilibrium point*

$E^*(A^*, P^*, H^*, E_P^*, R^*)$ where

$$A^* = \frac{(B_1+\delta_1 r)\mu + \beta_2 r^2\delta_1 + ((\delta_1+\delta_2)\alpha_1 + \alpha_3\delta_1 + \beta_2 B_1)r + (B_1+B_2)\alpha_1 + B_1\alpha_3}{(\beta_1 r + \alpha_2 + \mu)(\beta_2 r + \alpha_1 + \alpha_3 + \mu)}$$

$$P^* = \frac{B_2 + \delta_2 r}{\beta_2 r + \alpha_1 + \alpha_3 + \mu},$$

$$\begin{aligned}
H^* = &\big(((\beta_1+\beta_2)r + \alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2+\delta_3+\eta_1+\eta_2)\mu^3 + \big(\beta_1\beta_2 r^2 + ((\alpha_1+\alpha_3+\delta_2+\delta_3\\
&+\eta_1+\eta_2)\beta_1 + (\alpha_2+\delta_1+\delta_3+\eta_1+\eta_2)\beta_2)r - B_1\beta_1 + (\alpha_1+\alpha_3+\delta_1+\delta_2+\delta_3+\eta_1)\alpha_2\\
&+(\alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2+\delta_3)\eta_2 - B_2\beta_2 + (\delta_1+\delta_2+\delta_3+\eta_1)(\alpha_1+\alpha_3)\big)\mu^2 + \big((\delta_3+\eta_1\\
&+\eta_2)\beta_1\beta_2 r^2 + \big(((\delta_1+\delta_3+\eta_1)\alpha_2 + (\alpha_2+\delta_1+\delta_3)\eta_2)\beta_2 + ((\alpha_1+\alpha_3+\delta_2+\delta_3)\eta_2 + (\delta_3\\
&+\eta_1)\alpha_1 - (B_1+B_2)\beta_2 + (\delta_2+\delta_3+\eta_1)\alpha_3)\beta_1\big)r - (B_1\eta_2 + (B_1+B_2)\alpha_1 + B_1\alpha_3)\beta_1 + ((\delta_1\\
&+\delta_2+\delta_3+\eta_1)(\alpha_1+\alpha_3))\alpha_2 + ((\alpha_1+\alpha_3+\delta_1+\delta_2+\delta_3)\alpha_2 + (\delta_1+\delta_2+\delta_3)(\alpha_1+\alpha_3))\eta_2\\
&-(\alpha_2+\eta_2)B_2\beta_2\big)\mu + \beta_1\beta_2\delta_3\eta_2 r^2 + \big((\delta_1+\delta_3)\beta_2\alpha_2\eta_2 + (-(B_1+B_2)\beta_2\eta_2 + (\alpha_1\delta_3 + (\delta_2\\
&+\delta_3)\alpha_3)\eta_2)\beta_1\big)r - ((B_1+B_2)\alpha_1 + B_1\alpha_3)\beta_1\eta_2 + ((\delta_1+\delta_2+\delta_3)(\alpha_1+\alpha_3))\alpha_2\eta_2\\
&- B_2\beta_2\alpha_2\eta_2\big)/\beta_3(\beta_1 r + \alpha_1+\alpha_3+\mu)(\beta_1 r + \alpha_2 + \mu)(\eta_2+\mu),
\end{aligned}$$

$E_P{}^* = r$ (Appendix A),

$$R^* = \frac{\eta_1 r}{\eta_2 + \mu}$$

E^* exists only if

i. $$\begin{aligned}&\big(\beta_1\beta_2 r^2 + ((\alpha_1+\alpha_3+\delta_2+\delta_3+\eta_1+\eta_2)\beta_1 + (\alpha_2+\delta_1+\delta_3+\eta_1+\eta_2)\beta_2)r + (\alpha_1+\alpha_3\\
&+\delta_1+\delta_2+\delta_3+\eta_1)\alpha_2 + (\alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2+\delta_3)\eta_2 + (\delta_1+\delta_2+\delta_3+\eta_1)(\alpha_1+\alpha_3)\big)\\
&> (B_1\beta_1 + B_2\beta_2)\end{aligned}$$

ii. $$\begin{aligned}&\big(((\delta_3+\eta_1+\eta_2)\beta_1\beta_2 r^2 + (((\delta_1+\delta_3+\eta_1)\alpha_2 + (\alpha_2+\delta_1+\delta_3)\eta_2)\beta_2 + ((\alpha_1+\alpha_3+\delta_2\\
&+\delta_3)\eta_2 + (\delta_3+\eta_1)\alpha_1 + (\delta_2+\delta_3+\eta_1)\alpha_3)\beta_1)r + ((\delta_1+\delta_2+\delta_3+\eta_1)(\alpha_1+\alpha_3))\alpha_2\\
&+((\alpha_1+\alpha_3+\delta_1+\delta_2+\delta_3)\alpha_2 + (\delta_1+\delta_2+\delta_3)(\alpha_1+\alpha_3))\eta_2\big) > \big((B_1+B_2)\beta_1\beta_2 r\\
&+(B_1\eta_2 + (B_1+B_2)\alpha_1 + B_1\alpha_3)\beta_1 + (\alpha_2+\eta_2)B_2\beta_2)\big)\end{aligned}$$

iii. $$\begin{aligned}&\big(\beta_1\beta_2\delta_3\eta_2 r^2 + ((\delta_1+\delta_3)\beta_2\alpha_2\eta_2 + ((\alpha_1\delta_3 + (\delta_2+\delta_3)\alpha_3)\eta_2)\beta_1)r + ((\delta_1+\delta_2+\delta_3)\\
&(\alpha_1+\alpha_3))\alpha_2\eta_2\big) > \big((B_1+B_2)\beta_1\beta_2\eta_2 r + ((B_1+B_2)\alpha_1 + B_1\alpha_3)\beta_1\eta_2 + B_2\beta_2\alpha_2\eta_2\big)\end{aligned}$$

2.3. Computation of threshold quantity (R_0)

Here, threshold is computed to analyse the behaviour of the system. The threshold is an average number of newly infected pollutants affected by a single infectious pollutant during their entire lifetime of creating pollution. Here, if threshold is greater than one, it means the quantity of pollutants in environment exceeds the carrying capacity which decreases rain intensity. That makes environment is unstable. And if threshold is less than one, it is under control.

We find the threshold quantity (R_0) using the next generation matrix method (Diekmann *et al.*, 2010) follows as

$$F=\begin{bmatrix} \beta_1 A+\beta_2 P+\beta_3 H & 0 & \beta_1 E_P & \beta_2 E_P & \beta_3 E_P \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V=\begin{bmatrix} \delta_1+\delta_2+\delta_3+\eta_1+\mu & -\eta_2 & 0 & 0 & 0 \\ -\eta_1 & \eta_2+\mu & 0 & 0 & 0 \\ \beta_1 A-\delta_1 & 0 & \beta_1 E_P+\alpha_2+\mu & -\alpha_1 & 0 \\ \beta_2 P-\delta_2 & 0 & 0 & \beta_2 E_P+\alpha_1+\alpha_3+\mu & 0 \\ \beta_3 H-\delta_3 & 0 & -\alpha_2 & -\alpha_3 & \beta_3 E_P+\mu \end{bmatrix}$$

Finding F and V about E_0, we get

$$F(E_0)=\begin{bmatrix} \left(\begin{array}{l}\beta_1\left(\dfrac{B_1(\alpha_1+\alpha_3+\mu)+B_2\alpha_1}{(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}\right)+\beta_2\left(\dfrac{B_2}{\alpha_1+\alpha_3+\mu}\right) \\ +\beta_3\left(\dfrac{B_1\alpha_2(\alpha_1+\alpha_3+\mu)+B_2(\alpha_2(\alpha_1+\alpha_3)+\alpha_3\mu)+B_3(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}{\mu(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}\right)\end{array}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V(E_0)=\begin{bmatrix} \delta_1+\delta_2+\delta_3+\eta_1+\mu & -\eta_2 & 0 & 0 & 0 \\ -\eta_1 & \eta_2+\mu & 0 & 0 & 0 \\ \beta_1\left(\dfrac{B_1(\alpha_1+\alpha_3+\mu)+B_2\alpha_1}{(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}\right)-\delta_1 & 0 & \alpha_2+\mu & -\alpha_1 & 0 \\ \beta_2\left(\dfrac{B_2}{\alpha_1+\alpha_3+\mu}\right)-\delta_2 & 0 & 0 & \alpha_1+\alpha_3+\mu & 0 \\ \beta_3\left(\dfrac{\begin{array}{l}B_1\alpha_2(\alpha_1+\alpha_3+\mu)+B_2(\alpha_2(\alpha_1+\alpha_3)+\alpha_3\mu) \\ +B_3(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)\end{array}}{\mu(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}\right)-\delta_3 & 0 & -\alpha_2 & -\alpha_3 & \mu \end{bmatrix}$$

Here, V is non-singular matrix.

The threshold of the system is the spectral radius of matrix FV^{-1}

$$R_0=\frac{\left(\begin{array}{l}(\eta_2+\mu)\big((\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)B_3\beta_3+\big((\alpha_2(\alpha_1+\alpha_3)+\alpha_3\mu)\beta_3+(\alpha_2+\mu)\beta_2\mu+\beta_1\alpha_1\mu\big)B_2 \\ +\big((\alpha_2(\alpha_1+\alpha_3)+\alpha_3\mu)\beta_3+(\alpha_1+\alpha_3+\mu)\beta_1\mu\big)B_1\big)\end{array}\right)}{\mu\big((\delta_1+\delta_2+\delta_3+\mu)(\eta_2+\mu)+\eta_1\mu\big)(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)} \qquad (2)$$

If $R_0>1$ the environmental pollutants decrease rain intensity. Equivalently, it suggests that all the three population are contributing in concentration of environmental pollutants.

3. Stability analysis of the equilibria

In this section, the local and global stability analysis will be conversed for the proposed model.

3.1. Local stability analysis

Here, we deliberate the local stability analysis about the equilibrium points. The following Jacobian matrix is made from system (1).
If all the condition of Routh-Hurwitz criteria (Routh 1877) is satisfied using Jacobian matrix, the equilibrium point is locally asymptotically stable.

$$J=\begin{bmatrix} -\beta_1E_P-\alpha_2-\mu & \alpha_1 & 0 & -\beta_1H+\delta_1 & 0\\ 0 & -\beta_2E_P-\alpha_1-\alpha_3-\mu & 0 & -\beta_2H+\delta_2 & 0\\ \alpha_2 & \alpha_3 & -\beta_3E_P-\mu & -\beta_3H+\delta_3 & 0\\ \beta_1E_P & \beta_2E_P & \beta_3E_P & \begin{pmatrix}\beta_1A+\beta_2P+\beta_3H\\ -\delta_1-\delta_2-\delta_3-\eta_1-\mu\end{pmatrix} & \eta_2\\ 0 & 0 & 0 & \eta_1 & -\eta_2-\mu \end{bmatrix}$$

(3)

Theorem 1: The environmental pollutants free equilibrium point E_0 is locally asymptotically stable if $a_{44}>0$.

Proof: From (3) and taking $a_{11}=\alpha_2+\mu$, $a_{22}=\alpha_1+\alpha_3+\mu$, $a_{33}=\mu$,

$$a_{44}=\delta_1+\delta_2+\delta_3+\eta_1+\mu-\frac{\beta_1\left(B_1(\alpha_1+\alpha_3+\mu)+B_2\alpha_1\right)}{(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}-\frac{\beta_2B_2}{\alpha_1+\alpha_3+\mu}$$
$$-\frac{\beta_3\left(B_1\alpha_2(\alpha_1+\alpha_3+\mu)+B_2\left(\alpha_2(\alpha_3+\alpha_1)+\alpha_3\mu\right)+B_3(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)\right)}{\mu(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)},$$

$a_{55}=\eta_2+\mu$, we have

$$J_0=\begin{bmatrix} -a_{11} & \alpha_1 & 0 & -\frac{\beta_1\left(B_1(\alpha_1+\alpha_3+\mu)+B_2\alpha_1\right)}{(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}+\delta_1 & 0\\ 0 & -a_{22} & 0 & -\frac{\beta_2B_2}{\alpha_1+\alpha_3+\mu} & 0\\ \alpha_2 & \alpha_3 & -a_{33} & -\left(\frac{\beta_3\begin{pmatrix}B_1\alpha_2(\alpha_1+\alpha_3+\mu)+B_2\left(\alpha_2(\alpha_3+\alpha_1)+\alpha_3\mu\right)\\ +B_3(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)\end{pmatrix}}{\mu(\alpha_1+\alpha_3+\mu)(\alpha_2+\mu)}\right) & 0\\ 0 & 0 & 0 & -a_{44} & \eta_2\\ 0 & 0 & 0 & \eta_1 & -a_{55} \end{bmatrix}$$

The characteristics equation of the Jacobian matrix J_0 is

$$\lambda_1^5+x_1\lambda_1^4+x_2\lambda_1^3+x_3\lambda_1^2+x_4\lambda_1+x_5$$

where

$$x_1=a_{55}+a_{44}+a_{33}+a_{22}+a_{11}$$

$$x_2=-\eta_1\eta_2+a_{55}a_{44}+a_{55}a_{33}+a_{55}a_{22}+a_{55}a_{11}+a_{44}a_{33}+a_{44}a_{22}+a_{44}a_{11}+a_{33}a_{22}+a_{33}a_{11}+a_{22}a_{11}$$
$$=\left(-\eta_1\eta_2+a_{55}a_{44}\right)+\left(a_{55}+a_{44}\right)\left(a_{33}+a_{22}+a_{11}\right)+a_{33}\left(a_{22}+a_{11}\right)+a_{22}a_{11}$$

$$x_3=-\eta_1\eta_2a_{33}-\eta_1\eta_2a_{22}-\eta_1\eta_2a_{11}+a_{55}a_{44}a_{33}+a_{55}a_{44}a_{22}+a_{55}a_{44}a_{11}+a_{55}a_{33}a_{22}+a_{55}a_{33}a_{11}+a_{55}a_{22}a_{11}$$
$$+a_{44}a_{33}a_{22}+a_{44}a_{33}a_{11}+a_{44}a_{22}a_{11}+a_{33}a_{22}a_{11}$$
$$=\left(-\eta_1\eta_2+a_{55}a_{44}\right)\left(a_{33}+a_{22}+a_{11}\right)+\left(a_{55}+a_{44}\right)\left(a_{33}\left(a_{22}+a_{11}\right)+a_{22}a_{11}\right)+a_{33}a_{22}a_{11}$$

$$x_4 = -\eta_1\eta_2 a_{33}a_{22} - \eta_1\eta_2 a_{33}a_{11} - \eta_1\eta_2 a_{22}a_{11} + a_{55}a_{44}a_{33}a_{22} + a_{55}a_{44}a_{33}a_{11} + a_{55}a_{44}a_{22}a_{11} + a_{55}a_{33}a_{22}a_{11} + a_{44}a_{33}a_{22}a_{11}$$
$$= \left(-\eta_1\eta_2 + a_{55}a_{44}\right)\left(a_{33}\left(a_{22}+a_{11}\right) + a_{22}a_{11}\right) + \left(a_{55}+a_{44}\right)a_{33}a_{22}a_{11}$$
$$x_5 = -\eta_1\eta_2 a_{33}a_{22}a_{11} + a_{55}a_{44}a_{33}a_{22}a_{11}$$
$$= \left(-\eta_1\eta_2 + a_{55}a_{44}\right)a_{33}a_{22}a_{11}$$

Here, x_1, x_2, x_3, x_4 and $x_5 > 0$ if

i. $a_{44} > 0$ (4)

ii. $-\eta_1\eta_2 + a_{55}a_{44} > 0$

$$\Rightarrow a_{55}a_{44} > \eta_1\eta_2$$
$$\Rightarrow \left(\eta_2+\mu\right)\left(\delta_1+\delta_2+\delta_3+\eta_1+\mu - \frac{\beta_1\left(B_1\left(\alpha_1+\alpha_3+\mu\right)+B_2\alpha_1\right)}{\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)} - \frac{\beta_2 B_2}{\alpha_1+\alpha_3+\mu} - \frac{\beta_3\left(B_1\alpha_2\left(\alpha_1+\alpha_3+\mu\right)+B_2\left(\alpha_2\left(\alpha_3+\alpha_1\right)+\alpha_3\mu\right)+B_3\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)\right)}{\mu\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)}\right) > \eta_1\eta_2$$
$$\Rightarrow \left(\eta_2+\mu\right)\left(a+\eta_1\right) > \eta_1\eta_2$$

where $a = \delta_1+\delta_2+\delta_3+\mu - \frac{\beta_1\left(B_1\left(\alpha_1+\alpha_3+\mu\right)+B_2\alpha_1\right)}{\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)} - \frac{\beta_2 B_2}{\alpha_1+\alpha_3+\mu} - \frac{\beta_3\left(B_1\alpha_2\left(\alpha_1+\alpha_3+\mu\right)+B_2\left(\alpha_2\left(\alpha_3+\alpha_1\right)+\alpha_3\mu\right)+B_3\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)\right)}{\mu\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)}$

$$\Rightarrow \eta_2 a + \mu a + \eta_1\eta_2 + \eta_1\mu > \eta_1\eta_2$$
$$\Rightarrow \eta_2 a + \mu a + \mu\eta_1 > 0 \text{ if } a > 0$$
$$\Rightarrow a + \eta_1 > \eta_1 > 0$$
$$\Rightarrow a + \eta_1 > 0$$
$$\Rightarrow a_{44} > 0 \quad (5)$$

Hence, from (4) and (5) E_0 is locally asymptotically stable if $a_{44} > 0$.

Theorem 2: The interior equilibrium point E^* is locally asymptotically if $b_{44} > 0$, $A^* > \frac{\delta_1}{\beta_1}$, $P^* > \frac{\delta_2}{\beta_2}$ and $H^* > \frac{\delta_3}{\beta_3}$.

Proof: From (3) and taking $b_{11} = \beta_1 E_P^* + \alpha_2 + \mu$, $b_{22} = \beta_2 E_P^* + \alpha_1 + \alpha_3 + \mu$, $b_{33} = \beta_3 E_P^* + \mu$, $b_{44} = -\beta_1 A^* - \beta_2 P^* - \beta_3 H^* + \delta_1 + \delta_2 + \delta_3 + \eta_1 + \mu$, $b_{55} = \eta_2 + \mu$, we have

$$J^* = \begin{bmatrix} -b_{11} & \alpha_1 & 0 & -\beta_1 A^* + \delta_1 & 0 \\ 0 & -b_{22} & 0 & -\beta_2 P^* + \delta_2 & 0 \\ \alpha_2 & \alpha_3 & -b_{33} & -\beta_3 H^* + \delta_3 & 0 \\ \beta_1 E_P^* & \beta_2 E_P^* & \beta_3 E_P^* & -b_{44} & \eta_2 \\ 0 & 0 & 0 & \eta_1 & -b_{55} \end{bmatrix}$$

The characteristics equation of the Jacobian matrix J^* is

$$\lambda_2{}^5 + x_6\lambda_2{}^4 + x_7\lambda_2{}^3 + x_8\lambda_2{}^2 + x_9\lambda_2 + x_{10}$$

where

$$x_6 = b_{55} + b_{44} + b_{33} + b_{22} + b_{11}$$

$$\begin{aligned}
x_7 &= -\eta_1\eta_2 + b_{55}b_{44} + b_{55}b_{33} + b_{55}b_{22} + b_{55}b_{11} + \beta_1{}^2E_P^*A^* - \beta_1E_P^*\delta_1 + \beta_2{}^2E_P^*P^* - \beta_2E_P^*\delta_2 + \beta_3{}^2E_P^*H^* \\
&\quad - \beta_3E_P^*\delta_3 + b_{44}b_{33} + b_{44}b_{22} + b_{44}b_{11} + b_{33}b_{22} + b_{33}b_{11} + b_{22}b_{11} \\
&= \left(-\eta_1\eta_2 + b_{55}b_{44}\right) + \beta_1E_P^*\left(\beta_1A^* - \delta_1\right) + \beta_2E_P^*\left(\beta_2P^* - \delta_2\right) + \beta_3E_P^*\left(\beta_3H^* - \delta_3\right) \\
&\quad + \left(b_{55} + b_{44}\right)\left(b_{33} + b_{22} + b_{11}\right) + b_{33}\left(b_{22} + b_{11}\right) + b_{22}b_{11}
\end{aligned}$$

$$\begin{aligned}
x_8 &= b_{55}\beta_3{}^2E_P^*H^* - b_{55}\beta_3E_P^*\delta_3 + b_{44}b_{33}b_{22} + \beta_1E_P^*\alpha_1\beta_2P^* + \beta_3E_P^*\alpha_2\beta_1A^* + \beta_3E_P^*\alpha_3\beta_2P^* - \beta_1E_P^*\alpha_1\delta_2 \\
&\quad - \beta_3E_P^*\alpha_2\delta_1 - \beta_3E_P^*\alpha_3\delta_2 + \beta_1{}^2E_P^*A^*b_{33} + \beta_1{}^2E_P^*A^*b_{22} - \beta_1E_P^*\delta_1b_{33} - \beta_1E_P^*\delta_1b_{22} + \beta_2{}^2E_P^*P^*b_{33} \\
&\quad + \beta_2{}^2E_P^*P^*b_{11} - \beta_2E_P^*\delta_2b_{33} - \beta_2E_P^*\delta_2b_{11} + \beta_3{}^2E_P^*H^*b_{22} + \beta_3{}^2E_P^*H^*b_{11} - \beta_3E_P^*\delta_3b_{22} - \beta_3E_P^*\delta_3b_{11} \\
&\quad + b_{55}\beta_1{}^2E_P^*A^* - b_{55}\beta_1E_P^*\delta_1 + b_{55}\beta_2{}^2E_P^*P^* - b_{55}\beta_2E_P^*\delta_2 + b_{33}b_{22}b_{11} + b_{44}b_{33}b_{11} + b_{44}b_{22}b_{11} - \eta_1\eta_2b_{33} \\
&\quad - \eta_1\eta_2b_{22} - \eta_1\eta_2b_{11} + b_{55}b_{44}b_{33} + b_{55}b_{44}b_{22} + b_{55}b_{44}b_{11} + b_{55}b_{33}b_{22} + b_{55}b_{33}b_{11} + b_{55}b_{22}b_{11} \\
&= \left(-\eta_1\eta_2 + b_{55}b_{44}\right)\left(b_{33} + b_{22} + b_{11}\right) + \left(\beta_1A^* - \delta_1\right)E_P^*\left(\beta_1\left(b_{55} + b_{33} + b_{22}\right) + \beta_3\alpha_2\right) \\
&\quad + \left(\beta_2P^* - \delta_2\right)E_P^*\left(\beta_1\alpha_1 + \beta_2\left(b_{55} + b_{33} + b_{11}\right) + \beta_3\alpha_3\right) + \left(\beta_3H^* - \delta_3\right)E_P^*\beta_3\left(b_{55} + b_{22} + b_{11}\right) \\
&\quad + \left(b_{55} + b_{44}\right)\left(b_{33}\left(b_{22} + b_{11}\right) + b_{22}b_{11}\right)
\end{aligned}$$

$$\begin{aligned}
x_9 &= -b_{55}\beta_1E_P^*\alpha_1\delta_2 - b_{55}\beta_3E_P^*\alpha_2\delta_1 - b_{55}\beta_3E_P^*\alpha_3\delta_2 + b_{55}\beta_1{}^2E_P^*A^*b_{33} + b_{55}\beta_1{}^2E_P^*A^*b_{22} - b_{55}\beta_1E_P^*\delta_1b_{33} \\
&\quad - b_{55}\beta_1E_P^*\delta_1b_{22} + b_{55}\beta_2{}^2E_P^*P^*b_{33} + b_{55}\beta_2{}^2E_P^*P^*b_{11} - b_{55}\beta_2E_P^*\delta_2b_{33} - b_{55}\beta_2E_P^*\delta_2b_{11} \\
&\quad + b_{55}\beta_3{}^2E_P^*H^*b_{22} + b_{55}\beta_3{}^2E_P^*H^*b_{11} - b_{55}\beta_3E_P^*\delta_3b_{22} - b_{55}\beta_3E_P^*\delta_3b_{11} - \beta_3E_P^*\alpha_2\alpha_1\delta_2 - \beta_1E_P^*\alpha_1\delta_2b_{33} \\
&\quad - \beta_3E_P^*\alpha_2\delta_1b_{22} - \beta_3E_P^*\alpha_3\delta_2b_{11} + \beta_1{}^2E_P^*A^*b_{33}b_{22} - \beta_1E_P^*\delta_1b_{33}b_{22} + \beta_2{}^2E_P^*P^*b_{33}b_{11} - \beta_2E_P^*\delta_2b_{33}b_{11} \\
&\quad + \beta_3{}^2E_P^*H^*b_{22}b_{11} - \beta_3E_P^*\delta_3b_{22}b_{11} + \beta_3E_P^*\alpha_2\alpha_1\beta_2P^* + \beta_1E_P^*\alpha_1\beta_2P^*b_{33} + \beta_3E_P^*\alpha_2\beta_1A^*b_{22} \\
&\quad + \beta_3E_P^*\alpha_3\beta_2P^*b_{11} - \eta_1\eta_2b_{33}b_{22} - \eta_1\eta_2b_{33}b_{11} - \eta_1\eta_2b_{22}b_{11} + b_{55}b_{44}b_{33}b_{22} + b_{55}b_{44}b_{33}b_{11} + b_{55}b_{44}b_{22}b_{11} \\
&\quad + b_{55}b_{33}b_{22}b_{11} + b_{55}\beta_1E_P^*\alpha_1\beta_2P^* + b_{55}\beta_3E_P^*\alpha_2\beta_1A^* + b_{55}\beta_3E_P^*\alpha_3\beta_2P^* + b_{44}b_{33}b_{22}b_{11} \\
&= \left(-\eta_1\eta_2 + b_{55}b_{44}\right)\left(b_{33}\left(b_{22} + b_{11}\right) + b_{22}b_{11}\right) + \left(\beta_1A^* - \delta_1\right)E_P^*\left(\beta_1\left(b_{55}\left(b_{33} + b_{22}\right) + b_{33}b_{22}\right)\right. \\
&\quad \left. + \beta_3\alpha_2\left(b_{55} + b_{22}\right)\right) + \left(\beta_2P^* - \delta_2\right)E_P^*\left(\beta_1\alpha_1\left(b_{55} + b_{33}\right) + \beta_2\left(b_{55}\left(b_{22} + b_{11}\right) + b_{33}b_{11}\right)\right. \\
&\quad \left. + \beta_3\left(\alpha_1\alpha_2 + \alpha_3\left(b_{55} + b_{33}\right)\right)\right) + \left(\beta_3H^* - \delta_3\right)E_P^*\beta_3\left(b_{55}\left(b_{22} + b_{11}\right) + b_{22}b_{11}\right) + \left(b_{55} + b_{44}\right)b_{33}b_{22}b_{11}
\end{aligned}$$

$$\begin{aligned}
x_{10} &= b_{55}\beta_3E_P^*\alpha_2\alpha_1\beta_2P^* + b_{55}\beta_1E_P^*\alpha_1\beta_2P^*b_{33} + b_{55}\beta_3E_P^*\alpha_2\beta_1A^*b_{22} + b_{55}\beta_3E_P^*\alpha_3\beta_2P^*b_{11} \\
&\quad - b_{55}\beta_3E_P^*\alpha_2\alpha_1\delta_2 - b_{55}\beta_1E_P^*\alpha_1\delta_2b_{33} - b_{55}\beta_3E_P^*\alpha_2\delta_1b_{22} - b_{55}\beta_3E_P^*\alpha_3\delta_2b_{11} \\
&\quad + b_{55}\beta_1{}^2E_P^*A^*b_{33}b_{22} - b_{55}\beta_1E_P^*\delta_1b_{33}b_{22} + b_{55}\beta_2{}^2E_P^*P^*b_{33}b_{11} - b_{55}\beta_2E_P^*\delta_2b_{33}b_{11} \\
&\quad + b_{55}\beta_3{}^2E_P^*H^*b_{22}b_{11} - b_{55}\beta_3E_P^*\delta_3b_{22}b_{11} - \eta_1\eta_2b_{33}b_{22}b_{11} + b_{55}b_{44}b_{33}b_{22}b_{11} \\
&= \left(\eta_1\eta_2 + b_{55}b_{44}\right)b_{33}b_{22}b_{11} + \left(\beta_1A^* - \delta_1\right)E_P^*b_{55}b_{22}\left(\beta_1b_{33} + \beta_3\alpha_2\right) \\
&\quad + \left(\beta_2P^* - \delta_2\right)E_P^*b_{55}\left(\beta_1\alpha_1b_{33} + \beta_2b_{33}b_{11} + \beta_3\left(\alpha_1\alpha_2 + \alpha_3b_{11}\right)\right) + \left(\beta_3H^* - \delta_3\right)E_P^*b_{55}\beta_3b_{22}b_{11}
\end{aligned}$$

Here, x_6, x_7, x_8, x_9 and $x_{10} > 0$ if

i. $b_{44} > 0$ (6)

ii. $-\eta_1\eta_2 + b_{55}b_{44} > 0$

$$\Rightarrow b_{55}b_{44} > \eta_1\eta_2$$

$$\Rightarrow \left(\eta_2 + \mu\right)\left(-\beta_1A^* - \beta_2P^* - \beta_3H^* + \delta_1 + \delta_2 + \delta_3 + \eta_1 + \mu\right) > \eta_1\eta_2$$

$$\Rightarrow \left(\eta_2 + \mu\right)\left(b + \eta_1\right) > \eta_1\eta_2$$

where $b = -\beta_1 A^* - \beta_2 P^* - \beta_3 H^* + \delta_1 + \delta_2 + \delta_3 + \mu$

$\Rightarrow \eta_2 b + \mu b + \eta_1\eta_2 + \mu\eta_1 > \eta_1\eta_2$

$\Rightarrow \eta_2 b + \mu b + \mu\eta_1 > 0$ if $b > 0$

$$\Rightarrow b + \eta_1 > \eta_1 > 0$$
$$\Rightarrow b + \eta_1 > 0$$
$$\Rightarrow b_{44} > 0 \quad (7)$$

iii. $\beta_1 A^* - \delta_1 > 0 \Rightarrow A^* > \dfrac{\delta_1}{\beta_1}$ (8)

iv. $\beta_2 P^* - \delta_2 > 0 \Rightarrow P^* > \dfrac{\delta_2}{\beta_2}$ (9)

v. $\beta_3 H^* - \delta_3 > 0 \Rightarrow H^* > \dfrac{\delta_3}{\beta_3}$ (10)

Hence, from (6) to (10) E^* is locally asymptotically stable if $b_{44} > 0$, $A^* > \dfrac{\delta_1}{\beta_1}$, $P^* > \dfrac{\delta_2}{\beta_2}$ and $H^* > \dfrac{\delta_3}{\beta_3}$.

3.2. Global stability analysis

The global stability analysis has been established using Lyapunov function for the system.

Theorem 3: The environmental pollutants free equilibrium point E_0 is globally asymptotically stable.

Proof: Consider the Lyapunov function

$$L(t) = E(t) + R(t)$$

Then,

$$\begin{aligned} L'(t) &= E'(t) + R'(t) \\ &= \beta_1 AE + \beta_2 PE + \beta_3 HE - \delta_1 E - \delta_2 E - \delta_3 E - \eta_1 E + \eta_2 R - \mu E + \eta_1 E - \eta_2 R - \mu R \\ &= B_1 + \delta_1 E + \alpha_1 P - \alpha_2 A - \mu A + B_2 + \delta_2 E - \alpha_1 P - \alpha_3 P - \mu P + B_3 + \delta_3 E + \alpha_2 A + \alpha_3 P \\ &\quad - \mu H - \delta_1 E - \delta_2 E - \delta_3 E - \mu(E + R) \\ &= B_1 - \mu A + B_2 - \mu P + B_3 - \mu H - \mu(E + R) \end{aligned} \quad (11)$$

Now, one can see that $A \le \dfrac{B_1}{\mu}$, $P \le \dfrac{B_2}{\mu}$ and $H \le \dfrac{B_3}{\mu}$.

Then by (11),

$$\begin{aligned} L'(t) &\le B_1 - \mu\left(\frac{B_1}{\mu}\right) + B_2 - \mu\left(\frac{B_2}{\mu}\right) + B_3 - \mu\left(\frac{B_3}{\mu}\right) \\ &= -\mu(E + R) \le 0 \end{aligned}$$

Now, using LaSalle's Invariance Principle (La Salle, 1976), $\dfrac{dL}{dt} \le 0$ whereas $\dfrac{dL}{dt} = 0$ only if $E + R = 0.$

Hence, E_0 is globally asymptotically stable.

Theorem 4: The interior equilibrium point E^* is globally asymptotically stable.

Proof: Consider the Lyapunov function

$$L(t) = \frac{1}{2}\left[(A - A^*) + (P - P^*) + (H - H^*) + (E - E^*) + (R - R^*)\right]^2$$

Then,

$$L'(t)=\left[\left(A-A^*\right)+\left(P-P^*\right)+\left(H-H^*\right)+\left(E-E^*\right)+\left(R-R^*\right)\right]\left[A'+P'+H'+E'+R'\right]$$

$$=\left[\left(A-A^*\right)+\left(P-P^*\right)+\left(H-H^*\right)+\left(E-E^*\right)+\left(R-R^*\right)\right]$$
$$\left[B_1+B_2+B_3-\mu A-\mu P-\mu H-\mu E-\mu R\right]$$

$$=\left[\left(A-A^*\right)+\left(P-P^*\right)+\left(H-H^*\right)+\left(E-E^*\right)+\left(R-R^*\right)\right]$$
$$\left[\mu A^*+\mu P^*+\mu H^*+\mu E^*+\mu R^*-\mu A-\mu P-\mu H-\mu E-\mu R\right]$$

$$=-\left[\left(A-A^*\right)+\left(P-P^*\right)+\left(H-H^*\right)+\left(E-E^*\right)+\left(R-R^*\right)\right]^2\le 0$$

where $B_1+B_2+B_3=\mu A^*+\mu P^*+\mu H^*+\mu E^*+\mu R^*$

Hence, E^* is globally asymptotically stable.

4. Bifurcation analysis

To find the backward bifurcation, environmental pollutants compartment should be at least non-zero. On solving system (1) for E_P^*, we have

$$f\left(E_P^*\right)=AE_P^{*2}+BE_P^*+C=0 \tag{12}$$

where

$$\begin{aligned}A=&\left(\left(\beta_1+\beta_2\right)\beta_3+\beta_1\beta_2\right)\mu^3+\left(\left(\left(\alpha_2+\delta_1+\eta_1+\eta_2\right)\beta_2+\left(\alpha_1+\alpha_3+\delta_2+\eta_1+\eta_2\right)+\beta_1\right)\beta_3+\left(\delta_3\right.\right.\\&\left.\left.+\eta_1+\eta_2\right)\beta_1\beta_2\right)\mu^2+\left(\left(\left(-\left(B_1+B_2+B_3\right)\beta_1+\alpha_2\eta_1+\left(\alpha_2+\delta_1\right)\eta_2\right)\beta_2+\left(\left(\alpha_1+\alpha_3+\delta_2\right)\eta_2\right.\right.\right.\\&\left.\left.\left.+\left(\alpha_1+\alpha_3\right)\eta_1\right)\beta_1\right)B_3+\beta_1\beta_2\delta_3\eta_2\right)\mu-\left(B_1+B_2+B_3\right)\beta_1\beta_2\beta_3\eta_2\end{aligned}$$

$$\begin{aligned}B=&\beta_1+\beta_2+\beta_3)\mu^4+\left(\left(\alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2+\eta_1+\eta_2\right)\beta_3+\left(\alpha_2+\eta_1+\eta_2+\delta_1+\delta_3\right)\beta_2+\left(\alpha_1\right.\right.\\&\left.\left.+\alpha_3+\delta_2+\delta_3+\eta_1+\eta_2\right)\beta_1\right)\mu^3+\left(\left(-\left(B_1+B_3\right)\beta_1+\left(\alpha_1+\alpha_3+\delta_2\right)\alpha_2+\left(\alpha_1+\alpha_2+\alpha_3+\delta_1\right.\right.\right.\\&\left.+\delta_2\right)\eta_2+\left(\delta_1+\delta_2\right)\alpha_1+\alpha_3\delta_1-\left(B_2+B_3\right)\beta_2+\left(\alpha_1+\alpha_2+\alpha_3\right)\eta_1\right)\beta_3+\left(-\left(B_1+B_2\right)\beta_1+\alpha_2\eta_1\right.\\&\left.+\left(\alpha_2+\delta_1+\delta_3\right)\eta_2+\left(\delta_1+\delta_3\right)\alpha_2\right)\beta_2+\left(\left(\alpha_1+\alpha_3\right)\eta_1+\alpha_1\delta_3+\left(\delta_2+\delta_3\right)\alpha_3+\left(\alpha_1+\alpha_3+\delta_2\right.\right.\\&\left.\left.\left.+\delta_3\right)\eta_2\right)\beta_1\right)\mu^2+\left(\left(-\left(\left(B_2+B_3\right)\eta_2+\left(B_1+B_2+B_3\right)\alpha_2\right)\beta_2+\left(\alpha_1+\alpha_3\right)\alpha_2\eta_1-\left(\left(B_1+B_3\right)\eta_2\right.\right.\right.\\&\left.\left.+\left(B_1+B_2+B_3\right)\left(\alpha_1+\alpha_3\right)\right)\beta_1+\left(\alpha_3\delta_1+\left(\delta_1+\delta_2\right)\alpha_1+\left(\alpha_1+\alpha_3+\delta_2\right)\alpha_2\right)\eta_2\right)\beta_3+\left(-\left(B_1\right.\right.\\&\left.\left.\left.+B_2\right)\beta_1\eta_2+\left(\delta_1+\delta_3\right)\alpha_2\eta_2\right)\beta_2+\left(\alpha_1\delta_3+\left(\delta_2+\delta_3\right)\alpha_3\right)\beta_1\eta_2\right)\mu-\left(\left(B_1+B_2+B_3\right)\beta_2\alpha_2\eta_2+\left(\left(B_1\right.\right.\right.\\&\left.\left.\left.+B_2+B_3\right)\left(\alpha_1+\alpha_3\right)\right)\beta_1\eta_2\right)\beta_3\end{aligned}$$

$$C=\mu\left(\left(\eta_2+\mu\right)\left(\delta_1+\delta_2+\delta_3+\mu\right)+\eta_1\mu\right)\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)\left(1-R_0\right)$$

The coefficient A must be always positive and C should depend upon the value of R_0, if $R_0<1$ then C is positive and if $R_0>1$ then C is negative. For $A>0$, the positive result depends upon the sign of B and C. The equation (12) have two roots; from that one is positive and other is negative for $R_0>1$. Now, if $R_0=1$ then $C=0$ and we obtain a non-zero solution of equation (12) as $\frac{-B}{A}$ which is positive if and only if $B<0$. For $B<0$, there exists a positive interior equilibrium point for $R_0=1$ that means the equilibria continuously depends upon R_0, indicating that there exists an interval for R_0 which have two positive equilibria $I_0=\frac{-B-\sqrt{B^2-4AC}}{2A}$, $I_1=\frac{-B+\sqrt{B^2-4AC}}{2A}$. For, backward bifurcation putting the discriminant $B^2-4AC=0$ and then solving for the critical points of R_0 gives

$$R_C=1-\frac{B^2}{4A\mu\left(\left(\eta_2+\mu\right)\left(\delta_1+\delta_2+\delta_3+\mu\right)+\eta_1\mu\right)\left(\alpha_1+\alpha_3+\mu\right)\left(\alpha_2+\mu\right)}$$

. If $R_C < R_0$ then $B^2 - 4AC > 0$ and for the point of R_0 backward bifurcation exists such that $R_C < R_0 < 1$ (Khan et al. (2014), Wangari et al. (2016)).

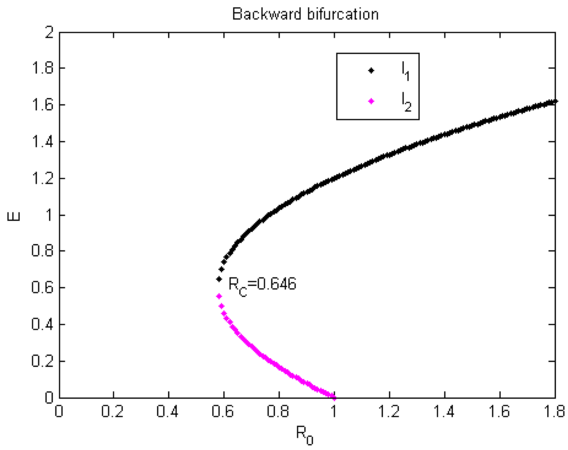

Figure 2: Backward bifurcation of environmental pollutants

5. Numerical simulations

In this section, numerical simulation is carried out to show analytical results for the transmission of environmental pollutants affecting rain intensity.

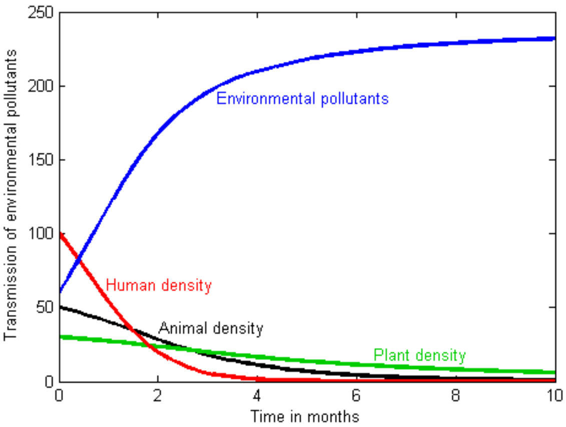

Figure 3: Transmission of environmental pollutants

It can be followed from figure 3 that animal consumed 20% plant produce, whereas human consumes 22% plant produce. But human also intakes 33% animal as a food which results reduction in the density of plants and animals. On the other hand, reverse effect is observed in environmental pollutants. Here, environmental pollutants increase by 74% due to three-layered population. Among that 82% environmental pollutants caused only due to human activities.

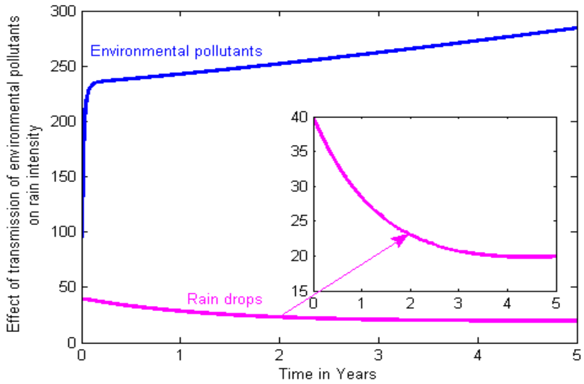

Figure 4: Effect of environmental pollutants on rain intensity

Figure 4 shows the effect of environmental pollutants on rain intensity. Environmental pollutants decrease rain intensity by 50% just in 5 years.

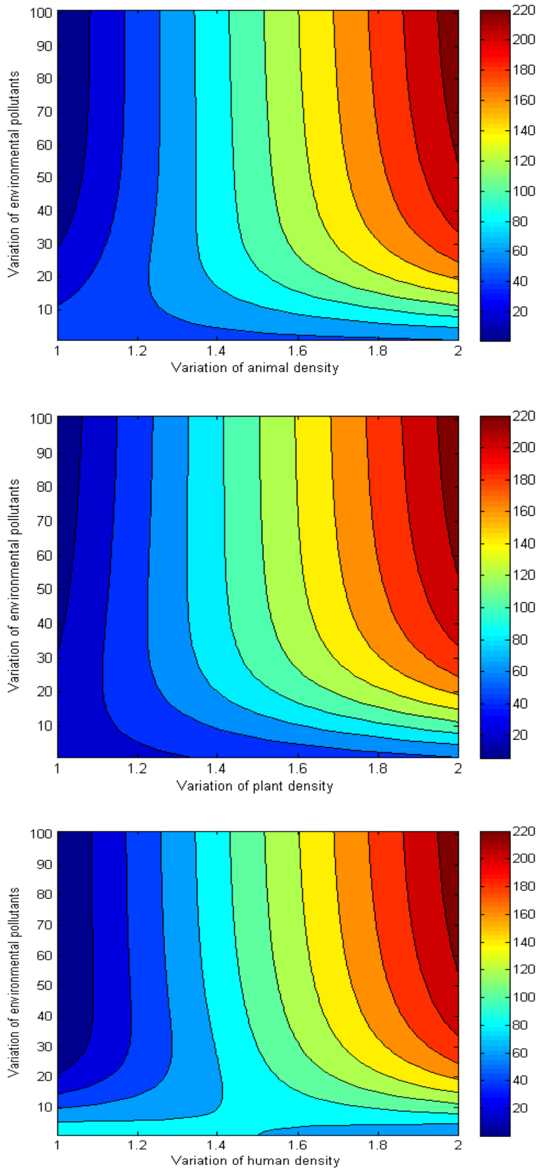

Figure 5: Environmental pollutants intensity due to three-layered population

40 PPM environmental pollution by animal population, 20 PPM by plant population and 80 PPM by human population turns out to be break-even point. Thereafter, animal, plant and human population increase environmental pollutants intensity drastically. This suggest that humans are the major source of environmental pollutants.

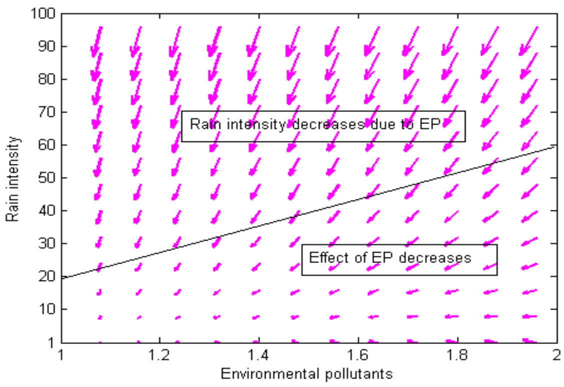

Figure 6: Behaviour of rain intensity due to environmental pollutants

Figure 6 suggests that environmental pollutants diminish rain intensity with the time. It can be observed that in the beginning, effect of environmental pollutants is high which reduces rain intensity rapidly. And with passage of time, intensity of environmental pollutants and rain decreases but at a lower rate.

Summary and conclusions

We formulated a mathematical model of bifurcation and stability analysis for environmental pollutants through three-layered population and rain. The proposed model shows the dynamics of environmental pollutants and consequently the rain intensity. Bifurcation analysis is computed through threshold quantity. This suggest that the existence of pollutants present in environment which affects rain intensity are under control.

The result shows that the model depends upon the value of the threshold quantity. After substituting the parametric values described in Table 1, the threshold quantity is intended as 0.7898 which advices that though situation is under control, government should put control as remedial steps.

Appendix A

The expression of r is given as follows:

$$\begin{aligned}
r = \text{root of } \{ & (\eta_1+\eta_2+\mu)\beta_1\beta_2\beta_3\mu Z^3 + [((\beta_1+\beta_2)\beta_3+\beta_1\beta_2)\mu^3 + (((\alpha_2+\delta_1+\eta_1+\eta_2)\beta_2 \\
& +(\alpha_1+\alpha_3+\delta_2+\eta_1+\eta_2)+\beta_1)\beta_3+(\delta_3+\eta_1+\eta_2)\beta_1\beta_2)\mu^2+(((-(B_1+B_2+B_3)\beta_1+ \\
& \alpha_2\eta_1+(\alpha_2+\delta_1)\eta_2)\beta_2+((\alpha_1+\alpha_3+\delta_2)\eta_2+(\alpha_1+\alpha_3)\eta_1)\beta_1)B_3+\beta_1\beta_2\delta_3\eta_2)\mu-(B_1 \\
& +B_2+B_3)\beta_1\beta_2\beta_3\eta_2]Z^2+[(\beta_1+\beta_2+\beta_3)\mu^4+((\alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2+\eta_1+\eta_2)\beta_3 \\
& +(\alpha_2+\eta_1+\eta_2+\delta_1+\delta_3)\beta_2+(\alpha_1+\alpha_3+\delta_2+\delta_3+\eta_1+\eta_2)\beta_1)\mu^3+((-(B_1+B_3)\beta_1 \\
& +(\alpha_1+\alpha_3+\delta_2)\alpha_2+(\alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2)\eta_2+(\delta_1+\delta_2)\alpha_1+\alpha_3\delta_1-(B_2+B_3)\beta_2 \\
& +(\alpha_1+\alpha_2+\alpha_3)\eta_1)\beta_3+(-(B_1+B_2)\beta_1+\alpha_2\eta_1+(\alpha_2+\delta_1+\delta_3)\eta_2+(\delta_1+\delta_3)\alpha_2)\beta_2 \\
& +((\alpha_1+\alpha_3)\eta_1+\alpha_1\delta_3+(\delta_2+\delta_3)\alpha_3+(\alpha_1+\alpha_3+\delta_2+\delta_3)\eta_2)\beta_1)\mu^2+((-((B_2 \\
& +B_3)\eta_2+(B_1+B_2+B_3)\alpha_2)\beta_2+(\alpha_1+\alpha_3)\alpha_2\eta_1-((B_1+B_3)\eta_2+(B_1+B_2+B_3)(\alpha_1 \\
& +\alpha_3))\beta_1+(\alpha_3\delta_1+(\delta_1+\delta_2)\alpha_1+(\alpha_1+\alpha_3+\delta_2)\alpha_2)\eta_2)\beta_3+(-(B_1+B_2)\beta_1\eta_2+(\delta_1 \\
& +\delta_3)\alpha_2\eta_2)\beta_2+(\alpha_1\delta_3+(\delta_2+\delta_3)\alpha_3)\beta_1\eta_2)\mu-((B_1+B_2+B_3)\beta_2\alpha_2\eta_2+((B_1+B_2 \\
& +B_3)(\alpha_1+\alpha_3))\beta_1\eta_2)\beta_3]Z+\mu^5+(\alpha_1+\alpha_2+\alpha_3+\delta_1+\delta_2+\delta_3+\eta_1+\eta_2)\mu^4+((\alpha_1 \\
& +\alpha_2+\alpha_3+\delta_1+\delta_2+\delta_3)\eta_2-(B_1\beta_1+B_2\beta_2+B_3\beta_3)+(\alpha_1+\alpha_2+\alpha_3)\eta_1+(\delta_1+\delta_2 \\
& +\delta_3)(\alpha_1+\alpha_3)+(\alpha_1+\alpha_3+\delta_1+\delta_2+\delta_3)\alpha_2)\mu^3+(-(\alpha_2+\eta_2)B_2\beta_2-((\alpha_1+\eta_2)B_3 \\
& +(B_1+B_3)\alpha_2+(B_2+B_3)\alpha_3)\beta_3+((\delta_1+\delta_2+\delta_3)(\alpha_1+\alpha_3)+(\alpha_1+\alpha_3+\delta_1+\delta_2 \\
& +\delta_3)\alpha_2)\eta_2-((\alpha_3+\eta_2)B_1+(B_1+B_2)\alpha_1)\beta_1+(\delta_1+\delta_2+\delta_3)(\alpha_1+\alpha_3)\alpha_2+(\alpha_1 \\
& +\alpha_3)\alpha_2\eta_1)\mu^2+((-B_3\alpha_1+(B_2+B_3)\alpha_3+(B_1+B_3)\alpha_2)\eta_2+(B_1+B_2+B_3)(\alpha_1 \\
& +\alpha_3))\alpha_2)\beta_3+((\delta_1+\delta_2+\delta_3)(\alpha_1+\alpha_3))\alpha_2\eta_2-(B_1\alpha_3+(B_1+B_2)\alpha_1)\beta_1\eta_2 \\
& -B_2\beta_1\alpha_2\eta_2)\mu-((B_1+B_2+B_3)(\alpha_1+\alpha_3))\beta_3\alpha_2\eta_2\}
\end{aligned}$$

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